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Unspanned Stochastic Covariations & Projective Geometry

Gregory Pelts

Wells Fargo

Quant Minds USA , Semptember 26, 2018

Abstract

Volatility is called unspanned if it can be dynamically separated from analytical representation of the underlying observables, such as swap or market rates. This quality is paramount for efficient calibration and pricing. Conformal symmetries provide a powerful tool for building parsimonious models of this kind. However, in this family of models, only common scale of volatility is unspanned. This limits the model calibration flexibility, particularly, in the low rate regime. Here, we demonstrate how to overcome these restrictions. This is achieved via application of projective geometry and abstract algebra.

Overview of Interest Rate Modeling Approaches

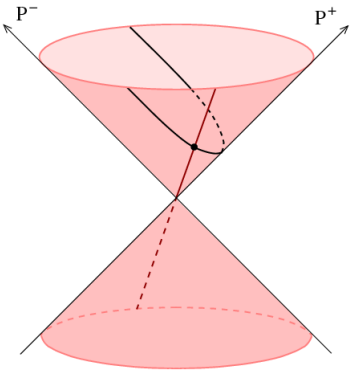
- Market models are too computationally expensive. In addition, they have limited predictive power due to their lack of structure and tractability.
- Unspanned stochastic volatility is virtually the only viable alternative. In this type of model, volatilities cannot be deduced from the yield curve alone, and may require additional state variables. From the risk-management perspective, this idea adheres to the common wisdom that one should not attempt to hedge swaptions or other rate derivatives with bonds and swaps alone. Analytically, this quality is paramount, as it enables bootstrapping calibration.
- Application of conformal symmetry provides a wide class of unspanned stochastic volatility models. However, the approach has the following limitations:
  - Only the common scale of volatility is unspanned; correlations and vol ratios are not.
  - Poor performance in the low rates regime. We have previously suggested to remedy the problem with a jump regime introduced via the stochastic time technique.
  - Although the model can be defined in any dimension $N \geq 3$, each version possess distinct properties. Lower dimensional models are not reduced versions of higher dimensional ones and, occasionally, may perform better. This imposes a hard limit on the number of risk factors the model can handle.
  - The model cannot be naturally extended to cover multiple currencies.

Conformal Symmetries



Conformal symmetry provides a powerful tool for building models that combine configurable, realistic dynamics and calibration flexibility with closed form formulae for most of the observables. Conformal maps may distort the image, but remain proportional in the small scale limit.

Conformal Symmetries and Lorentz Transformations



- Consider an $n+2$ dimensional *Minkowski Space* $\mathbb{T}$ with the hyperbolic metric

$$(x,x) = x_1^2 + \cdots + x_n^2 + x_{n+1}^2 - x_{n+2}^2.$$

- The *Light Cone* $\mathbb{T}_0 = \{x \,|\, (x,x) = 0\}$ inherits from $\mathbb{T}$, a positive semi-definite metric degenerate along the scaling dimension.
- The *Projective Light Cone* X (a set of light rays) inherits positive definite conformal metrics from $\mathbb{T}_0$.

Hyperbolic rotations (*Lorentz transformations*)

$$U: \quad \mathbb{T} \to \mathbb{T}, \quad (Ux, Ux) = (x,x)$$

induce conformal transformations in X.

Projective Light Cone in Flat Parameterization

Isotropic Basis

$$\begin{aligned}(x,x) &= x_1^2 + \cdots + x_n^2 + 2 \cdot \tfrac{1}{\sqrt{2}}(x_{n+1} + x_{n+2}) \cdot \tfrac{1}{\sqrt{2}}(x_{n+1} - x_{n+2}) \\ &= x_1^2 + \cdots + x_n^2 + 2x_+x_-\end{aligned}$$

- For each light ray $p \in X$, select the unique point with $x_+ = 1$.
- Then $(x,x) = 0$ iff $x_- = -\frac{1}{2}(x_1^2 + \cdots + x_n^2)$.
- The light ray based on the vector $e_- = (0, \ldots, 0, 0, 1)$, with the trivial x_+ component, is excluded in this parameterization. It can be associated with the *point of infinity*.

$\mathbb{R}^N$-Representation of Hyperbolic Rotations

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \sim \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \\ -\frac{x_1^2+\cdots+x_n^2}{2} \end{pmatrix} \overset{U}{\mapsto} \begin{pmatrix} x_1' \\ \vdots \\ x_n' \\ x_+' \\ x_-' \end{pmatrix} \cong \begin{pmatrix} x_1'/x_+' \\ \vdots \\ x_n'/x_+' \\ 1 \\ x_-'/x_+' \end{pmatrix} \sim \begin{pmatrix} x_1'/x_+' \\ \vdots \\ x_n'/x_+' \end{pmatrix}$$

Conformally Invariant Diffusion Generator

- The Laplacian $\Delta = \partial^2_{x_1} + \cdots + \partial^2_{x_N}$ is conformally invariant for $n = 2$ modulo scaling.
- There are no conformally invariant elliptic differential operators for $n > 2$.
- To realize the conformal invariance of the Laplacian, we should interpret it differently.
- Consider a space $\mathcal{F}^\gamma$ of homogeneous functions over the light cone:

$$\forall x \in \mathbb{T}_0,\ \lambda \in \mathbb{R}\colon\ \psi(\lambda x) = |\lambda|^\gamma \cdot \psi(x).$$

- Homogeneous functions of the order $\gamma = 0$ can be thought of as functions on the projective light cone:

$$F\colon\ X \to \mathbb{R}.$$

- Homogeneous functions of a nontrivial order can be mapped to ordinary light cone functions as follows:

$$\mathcal{F}^\gamma \ni \mathcal{F} = Fx_+^\gamma = F(e_-, x)^\gamma.$$

- Consider a linear differential operator:

$$\mathcal{F}^\gamma \xrightarrow{\diamondsuit} \mathcal{F}^{\gamma'}, \quad \diamondsuit Fx_+^\gamma = (\Delta F)x_+^{\gamma'}.$$

- It appears to be Lorentz invariant (basis independent) iff $\gamma = 1 - \frac{n}{2}$, $\gamma' = -1 - \frac{n}{2}$.

Wealth Bundle and Numeraire-Free Framework

- The conformally invariant generator operates on the space $\mathcal{F}^\gamma$, which is different from the space of ordinary functions.
- We can map elements of this space to ordinary functions by choosing a unit element, e.g., x_+^γ.
- The backward Kolmogorov equation operates on value functions.
- Values of financial products are not just numbers, they also have units.
- There is no universal unit. Any positive self-financing account can serve as a numeraire.
- Mathematically, functions taking values measured in units can be interpreted as sections of a line bundle.
- The line bundle associated with financial products will be called the wealth bundle.
- A ratio of two sections is unitless. It can be interpreted as an exchange rate of the associated assets.
- The nontrivial value of γ reflects the fact that conformal symmetries are accompanied by changes in numeraire.

Discount Factors, Forward Measure And Parameterizations

- Let us choose, as a numeraire, a zero coupon bond expiring at $T > t$.
- The value of the bond itself in this numeraire is given by the constant $\mathbf{1}$-function, which is obviously harmonic.
- It corresponds to a *harmonic wealth section*

$$\mathcal{F}^\gamma \ni \mathcal{B}_T = x_+^\gamma = (e_-, x)^\gamma, \quad \Diamond \mathcal{B}_T = 0.$$

- By symmetry, we can associate a similar harmonic section with any light cone vector:

$$\forall b \in \mathbb{T}_0 : \quad \Diamond (b, x)^\gamma = 0.$$

Indeed, in the flat parameterization, such sections are associated with functions known to be harmonic:

$$(b, x)^\gamma = (b_+ x_+)^\gamma |\mathbf{b} - \mathbf{x}|^{2\gamma} = \left(\sqrt{b_+ x_+} |\mathbf{b} - \mathbf{x}| \right)^{2-n}.$$

Here:

$$\mathbf{b} = \left(\frac{b_1}{b_+}, \dots, \frac{b_n}{b_+} \right), \quad \mathbf{x} = \left(\frac{x_1}{x_+}, \dots, \frac{x_n}{x_+} \right).$$

- It is natural to assume that they can be related to the zero-coupon bonds of other expirations:

$$\forall T > t : \quad \mathcal{B}_T(x) = (b_T, x)^\gamma.$$

The Process

- Under our assumptions, forward discount factors can be calculated as:

$$B(T,T') = \frac{\mathcal{B}_{T'}}{\mathcal{B}_T} = \left(\frac{(b_{T'},x)}{(b_T,x)}\right)^{\gamma}.$$

- They satisfy the required martingale property as long as the $\mathbf{x}$-process in the forward T-measure is of the form

$$d\mathbf{x}_i = \sigma dW_i.$$

Here W_i are independent Wiener processes.

Unspanned Property

There are virtually no restrictions on the σ-process. It might be fully subordinated to the $\mathbf{x}$-process $\sigma = \sigma(\mathbf{x},\mathbf{t})$, or governed by any number of additional stochastic factors. The choice of the σ-process does not influence the yield curve. This property is particularly important for bootstrapping calibration to swaptions.

Time Homogeneity and Hyperbolic (Lorentz) Evolution

- The forward parametrization is not stationary. To enable time homogeneity, light cone vectors b_T associated with different expiration should be related via some hyperbolic evolution:

$$b_{T'} = e^{k(T'-T)} U_{T'-T} b_T \quad (U_\tau = e^{\epsilon\tau}, k \in \mathbb{R}).$$

  Here, ϵ is an infinitesimal hyperbolic rotation. Similarly to infinitesimal orthogonal rotations, it can be associated with a skew-symmetric form:

$$\forall x, y: \quad (x, \epsilon y) + (y, \epsilon x) = 0.$$

- The choice of ϵ is central to the model. It determines the parametric form of future yield curves and the correlation structure of forward rates. It is calibrated to swaptions with short expirations.

Stationary Parameterization

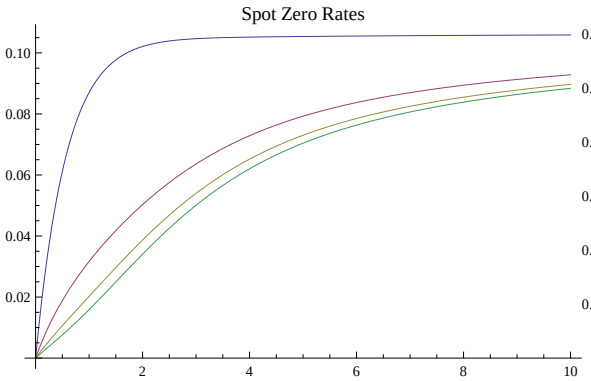
The process can be made explicitly time homogeneous in the following *stationary* parameterization:

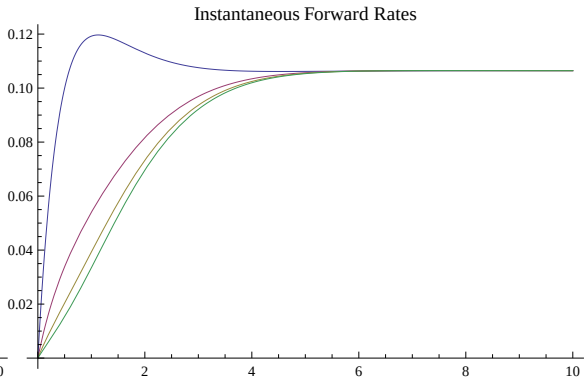
$$\tilde{\mathbf{x}} = U_{t-T}\mathbf{x}.$$

In this parameterization, the instantaneous spot rate can be related to one of the $\tilde{\mathbf{x}}$-variables:

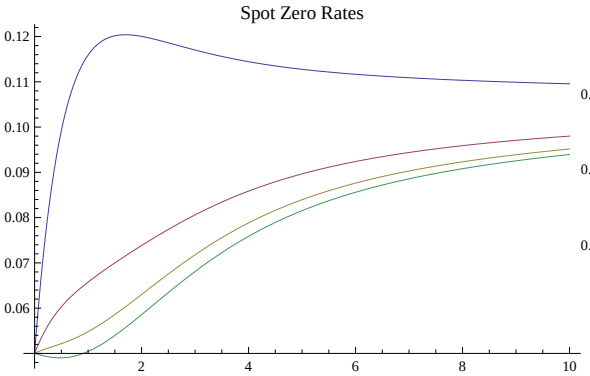
$$r = \tilde{\mathbf{x}}_1.$$

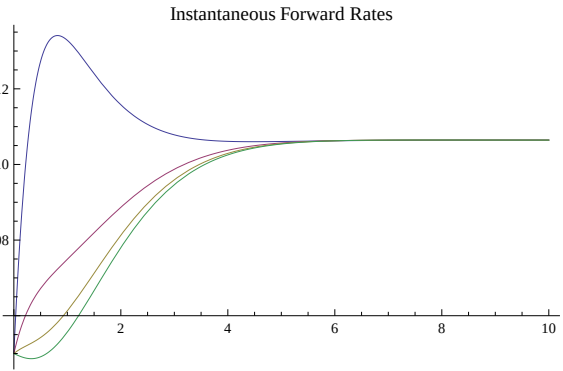
Conformal Yield Curve (Zero Short Rate)





Conformal Yield Curve (5% Short Rate)





Conformal Symmetry in $2D$ and Complex Geometry

In the $2D$ version of the conformal model, the Laplace equation is conformally invariant in the space of ordinary functions. From a financial perspective, this indicates the existence of the universal conformal invariant numeraire associated with all the T-measures. Hence, this version of the models is not applicable to stochastic interest rates.

Conformal $2D$ symmetries can also be associated with holomorphic transformations. Therefore, the state space of the model can be interpreted as a Riemann surface, i.e., a $1D$ complex manifold. Any compact Riemann surface of trivial topology is isomorphic to the *Riemann sphere* (compactified complex plain) $\bar{\mathbb{C}}$. The Riemann sphere is isomorphic to the complex projective line $\mathbb{CP}^1$:

$$\mathbb{C} \ni z \to \left[\begin{pmatrix} 1 \\ z \end{pmatrix}\right] \in \mathbb{CP}^1, \quad \infty \to \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \in \mathbb{CP}^1.$$

Hereafter, $[\cdot]$ denotes the equivalence class by scaling. The complex projective line $\mathbb{CP}^1$ is a representation of the $2D$ linear group $SL(2,\mathbb{C})$ quotient by its central subgroup $\{\mathbb{I}, -\mathbb{I}\}$.

Relations to the Light Cone Representation

Consider a space of 2×2 complex hermitian matrices:

$$\mathfrak{h}(2,\mathbb{C}) \equiv \left\{ h \in \mathbb{C}^2 \otimes \mathbb{C}^2 : \ h^* \equiv \bar{h}^T = h \right\}.$$

It is equipped with an invariant hyperbolic quadratic form defined as:

$$(h,h) \equiv -\det(h) = |h_{12}|^2 - h_{11}h_{22}, \quad h = \begin{pmatrix} h_{11} & h_{12} \\ h_{12}^* & h_{22} \end{pmatrix} \in \mathbb{T}, \quad h_{11} \in \mathbb{R}, \ h_{22} \in \mathbb{R}, \ h_{12} \in \mathbb{C}.$$

Hence, it can serve as Minkowski space $\mathbb{T} = \mathfrak{h}(2,\mathbb{C})$. Light cone vectors $(h,h) = 0$ are associated with singular matrices. The latter can be represented as $h = \xi \otimes \xi^* \ \left(\xi \in \mathbb{C}^2\right)$. This establishes a map from $\mathbb{C}^2/U(1)$ to the light cone:

$$\mathbb{C}^2 \ni \xi \to \xi \otimes \xi^* \in \mathbb{T}_0.$$

This, in turn, induces an isomorphism between $\mathbb{CP}^1$ and the projective light cone X.

Multidimensional Holomorphic Models

The real portion of any holomorphic function is a local martingale, as long as the diffusion generator is of the form:

$$\mathcal{D} = \tfrac{1}{2}\sigma\Delta = \tfrac{1}{2}\partial_z\partial_{\bar{z}}$$

Here, σ is an arbitrary (unspanned) stochastic volatility process.

The construct can be extended to an arbitrary complex dimension. Let f be a holomorphic function on K-dimensional complex manifold. Then $\Re(f)$ is a local martingales for any diffusion with a generator

$$\mathcal{D} = \tfrac{1}{2}\sum_{i,j}\sigma^{ij}\partial_{z^i}\partial_{\bar{z}_i}, \quad \sigma \in \mathfrak{h}(K,\mathbb{C}), \quad \dim_{\mathbb{R}}\mathfrak{h}(K,\mathbb{C}) = K^2.$$

Here, $\{z_i\}_{i=1}^{K}$ is a set of complex variables parameterizing the manifold, and σ_{ij} is an arbitrary positive definite hermitian matrix. All of the K^2 independent σ-components are arbitrary stochastic processes. Hence, in this model, besides the common volatility scale, some individual volatilities and correlation are separately unspanned. The total dimension of symmetric $2K \times 2K$ matrices is $K(2K+1)$.

The projective space $\mathbb{CP}^K$ is an example of a compact K-dimensional complex manifold we can use in the model. It is symmetric under the group $SL(K+1,\mathbb{C})$ of linear transformations in $\mathbb{C}^{K+1}$.

4D Conformal Models and Quaternions

Multidimensional holomorphic models preserve certain distinct characteristics of the $2D$ conformal model they extend. In particular, the diffusion generator operates on ordinary functions. Hence, the holomorphic models are unsuitable for stochastic interest rates.

The group of $4D$ orthogonal rotation $SO(4,\mathbb{R})$ has an amazing property. It can be decomposed to a direct product of a pair of $SO(3,\mathbb{R})$:

$$SO(4,\mathbb{R}) \cong SO(3,\mathbb{R}) \times SO(3,\mathbb{R}).$$

First, let us compare dimensions:

$$\dim SO(4,\mathbb{R}) = \frac{4 \cdot 3}{2} = 6 = 2 \dim SO(3,\mathbb{R}).$$

The relations can be elegantly explained via *quaternions*. Quaternions $\mathbb{H}$ are extensions of complex numbers $\mathbb{C}$, in a way similar to which $\mathbb{C}$ extends $\mathbb{R}$. As the name suggests $\dim_{\mathbb{R}} \mathbb{H} = 4$.

Quaternions - Brief Introduction

Quaternions were first introduced by William Rowan Hamilton in 1843. They are generally represented in the form:

$$z = a + b\mathbb{i} + c\mathbb{j} + d\mathbb{k}$$

with the following multiplication table

$\times$	$\mathbb{i}$	$\mathbb{j}$	$\mathbb{k}$
$\mathbb{i}$	-1	$\mathbb{k}$	$-\mathbb{j}$
$\mathbb{j}$	$-\mathbb{k}$	-1	$\mathbb{i}$
$\mathbb{k}$	$\mathbb{j}$	$-\mathbb{i}$	-1

Similarly to $\mathbb{R}$ and $\mathbb{C}$, quaternions are associative:

$$(z_1 z_2)z_3 = z_1(z_2 z_3).$$

However, they are not commutative.

Quaternions (cont)

Analogously to $\mathbb{C}$, quaternions are equipped with the conjugation:

$$z = a + b\mathbb{i} + c\mathbb{j} + d\mathbb{k} \to \bar{z} = z^* = a - b\mathbb{i} - c\mathbb{j} - d\mathbb{k}.$$

The conjugation is an anti-involution:

$$\bar{z}^* = z, \quad (z_1 z_2)^* = z_2^* z_1^*.$$

Similarly to the complex numbers, conjugation can be used to separate *real* and *imaginary* parts of a quaternion:

$$\Re z = \tfrac{1}{2}(z + \bar{z}), \quad \Im z = \tfrac{1}{2}(z - \bar{z}).$$

Hence the anti-evolution property, $z\bar{z} \in \mathbb{R}$. In addition:

$$z\bar{z} = (a + b\mathbb{i} + c\mathbb{j} + d\mathbb{k})(a - b\mathbb{i} - c\mathbb{j} - d\mathbb{k}) = a^2 + b^2 + c^2 + d^2 \equiv |z|^2 > 0 \ (z \neq 0).$$

Quaternions (cont)

Quaternion norm is multiplicative

$$|z_1 z_2| = |z_1|\,|z_2|\,.$$

Indeed

$$|z_1 z_2|^2 = z_1 z_2 \,(z_1 z_2)^* = z_1 z_2 \bar{z}_2 \bar{z}_1 = z_1 \,|z_2|^2\, \bar{z}_1 = |z_1|^2\,|z_2|^2\,.$$

Hence, the reciprocal can be defined as

$$\forall_{z\neq 0} : \exists z^{-1} = \frac{1}{|z|^2} z.$$

Algebras with such properties are called normed devision (or Hurwitz) algebras. There exist only four distinct finite dimensional Hurwitz algebras: real numbers $\mathbb{R}$ ($\dim \mathbb{R} = 1$), complex numbers $\mathbb{C}$ ($\dim \mathbb{C} = 2$), quaternions $\mathbb{H}$ ($\dim \mathbb{H} = 4$), and octonions $\mathbb{O}$ ($\dim \mathbb{O} = 8$). As we shall see, similarly to $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ can be used to extend the $4D$ and $8D$ conformal models, respectively.

Quaternions (cont)

A union quaternion is a quaternion of norm one. Union quaternions form a compact three dimensional group by multiplication. We shall denote it as $U(1,\mathbb{H})$. There are three distinct linear orthogonal representations of this group in $\mathbb{H}$:

- by left multiplication: $z \to uz$
- by right multiplication: $z \to z\bar{u}$
- by mixed multiplication: $z \to uz\bar{u}$

The last one stabilizes real elements of $\mathbb{H}$. Therefore, it also defines an orthogonal representation of $U(1,\mathbb{H})$ in the $3D$ space of imaginary quaternions. Thus we demonstrated that

$$SO(3,\mathbb{R}) \cong U(1,\mathbb{H})/\{1,-1\}\,.$$

The *chiral* representations commute with each other. Together, they form a central extension of the group of orthogonal rotations in $\mathbb{H}$:

$$SO(4,\mathbb{R}) = U(1,\mathbb{H}) \times U(1,\mathbb{H})/\{(1,1),(-1,-1)\}\,.$$

Quaternionic Projective Spaces

Complex numbers are related to $2D$ conformal geometry. Similarly, quaternions are related to $4D$ conformal geometry. Namely, $4D$ conformal symmetries can be associated with analytical quaternionic transformations. Due to the lack of commutativity, the accurate generic definition of a quaternionic (*Kähler*) manifold is rather complicated. For our purposes, it will suffice to explore a special case of quaternionic projective spaces $\mathbb{HP}^K$.

Consider a group $GL(K+1,\mathbb{H})$ of nonsingular $K+1 \times K+1$ quaternionic matrices. Due to the lack of commutativity, its representation in $\mathbb{H}^K$ is not linear w.r.t. the left multiplication. However, it is linear w.r.t. the right multiplication. Hence, $GL(K+1,\mathbb{H})$ has a natural representation in the projective space defined as

$$\mathbb{HP}^K \equiv \mathbb{H}^{K+1}/(\cdot\mathbb{H}).$$

Unlike for commutative algebras, the center of $GL(K+1,\mathbb{H})$ contains only real multiplications, rather than the whole of $\mathbb{H}$. In addition, there is no quaternionic determinant, so we have to get by with the real one, and define the special subgroup of $GL(K+1,\mathbb{H})$ as

$$SL(K+1,\mathbb{H}) = \left\{ \mathfrak{G} \in GL(K+1,\mathbb{H}) | \det_{\mathbb{R}}(\mathfrak{G}) = 1 \right\}, \quad \dim_{\mathbb{R}} SL(K+1,\mathbb{H}) = 4(K+1)^2 - 1.$$

Quaternionic Projective Line and Conformal Symmetries in $4D$

As usual, the quaternionic projective line $\mathbb{HP}^1$ is isomorphic to the compactified quaternions:

$$\mathbb{H} \ni z \to \left[\begin{pmatrix} 1 \\ z \end{pmatrix}\right] \in \mathbb{HP}^1, \quad \infty \to \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \in \mathbb{HP}^1.$$

In this parametrization, $GL(2,\mathbb{H})$ representation operates as

$$\mathbb{H} \ni z \to \begin{pmatrix} 1 \\ z \end{pmatrix} \xrightarrow{\mathfrak{G}} \begin{pmatrix} \mathfrak{G}_{11} + \mathfrak{G}_{12} z \\ \mathfrak{G}_{21} + \mathfrak{G}_{22} z \end{pmatrix} \to (\mathfrak{G}_{11} + \mathfrak{G}_{12} z)(\mathfrak{G}_{21} + \mathfrak{G}_{22} z)^{-1}.$$

These transformation are conformal in $\mathbb{H}$.

The conformal transformation in $4D$ can also be associated with 6-dimensional Lorentz rotations. First, let us check the dimensions:

$$\dim_{\mathbb{R}} SL(2,\mathbb{H}) = 4 \cdot 2^2 - 1 = 15, \quad \dim SO(5,1) = \frac{6 \cdot 5}{2} = 15.$$

Quaternionic Projective Line and Conformal Symmetries in $4D$ (cont.)

Consider the space of $2D$ quaternionic hermitian matrices

$$\mathfrak{h}(2,\mathbb{H}) \ni h = \begin{pmatrix} h_{11} & h_{12} \\ h_{12}^* & h_{22} \end{pmatrix} \quad (h_{11} \in \mathbb{R},\ h_{22} \in \mathbb{R},\ h_{12} \in \mathbb{H}).$$

Similarly to $\mathfrak{h}(2,\mathbb{C})$, this space is equipped with a hyperbolic quadratic form

$$(h,h) \equiv -\det(h) = |h_{12}|^2 - h_{11}h_{22}.$$

Hence, it can serve as a Minkowski space $\mathbb{T} = \mathfrak{h}(2,\mathbb{H})$. Like before, light cone vectors are associated with singular matrices. The latter can be represented as $h = \xi \otimes \xi^*$ $\left(\xi \in \mathbb{H}^2\right)$. This establishes a map from $\mathbb{H}^2/U(1,\mathbb{H})$ to the light cone:

$$\mathbb{H}^2 \ni \xi \to \xi \otimes \xi^* \in \mathbb{T}_0$$

leading to the isomorphism between $\mathbb{HP}^1$ and the projective light cone X.

Asset Representation in Quaternionic Form

Asset prices in the model are associated with the homogeneous functions of the order $\gamma = 1 - \frac{4}{2} = -1$. In the quadraphonic form, they correspond to the homogeneous functions on $\mathbb{H}^2$ of the order -2:

$$\psi : \mathbb{H}^2 \to \mathbb{R}, \quad \psi(\xi\lambda) = |\lambda|^{-2}\,\psi(\xi) \quad \left(\xi \in \mathbb{H}^2,\ \lambda \in \mathbb{H}\right).$$

Multiplication by unit quaternions does not affect the associated light cone vector:

$$(\xi\lambda) \times (\xi\lambda)^* = (\xi\lambda) \times \bar{\lambda}\xi^* = |\lambda|^2\, \xi \times \xi^* = \xi \times \xi^*.$$

Homogeneous functions of the form $|\langle \eta, \xi \rangle|^{-2}$ solve the backward conformal Kolmogorov equation. Here, η is a quaternionic linear form in $\mathbb{H}^2$:

$$\langle \eta, \xi \rangle = \eta_1 \xi^1 + \eta_2 \xi^2.$$

These solutions are used to represent zero-coupon bonds:

$$\mathcal{B}_T(\xi) = |\langle \eta_T, \xi \rangle|^{-2}.$$

Evolution, Discount Curve, and Dynamics

In the quaternionic form, the $4D$ conformal evolution is related to a linear evolution in $\mathbb{H}^2$:

$$\eta_T = \eta_t \exp(-\epsilon(T-t)) \quad \left(\epsilon \in \mathbb{H}^2 \otimes \mathbb{H}^2\right).$$

Bonds are associated with linear forms in $\mathbb{H}^2$. The representation of $SL(2,\mathbb{H})$ in the space of linear forms (row vectors) is dual to the one in $\mathbb{H}^2$ (column vectors). Unlike the latter, matrices operate on linear forms from the right side, while scalar multiplication is left-sided. Choose a linear form associated with cash as

$$\eta_t = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Then the discount factors can be calculated as

$$B(t,T) = \begin{pmatrix} 0 & 1 \end{pmatrix} \exp(-\epsilon(T-t)) \begin{pmatrix} 1 \\ z \end{pmatrix} \quad (z \in \mathbb{H}).$$

In the cash numeraire, the conformal diffusion generator is of the form

$$\mathcal{D} \approx \tfrac{1}{2}\sigma\Delta = \tfrac{1}{2}\sigma\partial_z\partial_{\bar{z}}.$$

Here, ∂_z and $\partial_{\bar{z}}$ denote *analytic quaternionic derivative* and its conjugate, respectively.

Multidimensional Quaternionic Models

Application of complex geometry lets us extend the $2D$ conformal model to a family of higher dimensional holomorphic models. Similarly, we can now extend the $4D$ model via application of quaternionic geometry. Associate states of the extended model with a quaternionic projective space $\mathbb{HP}^K$. Assuming a nontrivial first component, the projective space can be parameterized by K quaternions

$$\mathbb{H}^K \ni \begin{pmatrix} z^1 \\ \vdots \\ z^K \end{pmatrix} \to \left[\begin{pmatrix} 1 \\ z^1 \\ \vdots \\ z^K \end{pmatrix} \right] \in \mathbb{HP}^K.$$

Associate asset values with homogeneous functions in $\mathbb{H}^{K+1}$ of the order -2:

$$\psi : \ \mathbb{H}^{K+1} \to \mathbb{R}, \quad \psi(\xi\lambda) = |\lambda|^{-2}\,\psi(\xi) \quad \left(\xi \in \mathbb{H}^{K+1},\ \lambda \in \mathbb{H}\right)$$

with the bond functions of the form:

$$\mathcal{B}_T(\xi) = |\langle \eta_t \exp(-\epsilon(T-t)\xi\rangle|^{-2}, \quad \left(\epsilon \in \mathbb{H}^{K+1} \otimes \mathbb{H}^{K+1}, \quad \eta_t = (\underbrace{0 \ \cdots \ 0}_{K}\ 1)\right).$$

Multidimensional Quaternionic Models (cont.)

The bond functions defined in this way solve the model dynamics as long as it is driven by a hermitian quaternionic covariation matrix $\sigma \in \mathfrak{h}(2, \mathbb{H})$, i.e., with the diffusion generator of the form

$$\mathcal{D} = \frac{1}{2} \sum_{i,j} \sigma^{ij} \partial_{z^i} \partial_{\bar{z}^j}.$$

This leaves us with

$$\dim \mathfrak{h}(2, \mathbb{H}) = (2K - 1)K$$

unspanned factors for configuring volatilities and correlations. The dimension of all the symmetric $2K \times 2K$ matrices is equal to $2K(4K + 1)$. The component of σ can follow an arbitrary set of stochastic processes. The choice of the evolution generator ϵ still, to a certain degree, determines the term rates shape and covariation structure. However, this determination is no longer complete and can be further adjusted during bootstrapping. Also, a vanishing volatility of short term rates no longer imposes vanishing volatilities in longer terms. This is particularly important for modeling of the low rate regime.

Muti-Currency Models

This extension provides a unified approach for modeling multiple currencies, incorporating their stochastic interest rates, foreign exchange rates and volatilities. Associate each modeled currency with a quaternionic linear form η_i $(i = 0, \ldots, L-1)$. Formally, we can do the same without the extension. However, the 4 state variables would not be sufficient to cover all the foreign exchange rates and interest rates, even for two currencies. In addition, vanishing short rate volatility of a single currency would completely *freeze* the entire market, including all the foreign exchange rates and the rest of interest rates.

The extended $\mathbb{HP}^K$-model of the quaternionic dimension $K \geq L$ overcomes these limitations. Consider

$$\eta_i{}^k = \delta_i^k \equiv \begin{cases} 1 & (i = k) \\ 0 & (i \neq k) \end{cases} .$$

Then, the foreign exchange rates will be determined by the first $L-1$ quaternionic variables $\{z_i\}_{i=1}^{L-1}$. The remaining quaternionic variables can be used for stochastic rates and volatilities.

Octonions and 8-Dimensional Conformal Symmetries

The extensions of the $2D$ and $4D$ models are based on the complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$. Quaternions can be further extended to *octonions* $\mathbb{O}$ ($\dim \mathbb{O} = 8$). Octonions were first discovered by John T. Graves in 1843 and independently by Arthur Cayley. Octonions are neither commutative nor associative. However, they satisfy a weaker form of associativity. Namely, any subalgebra in $\mathbb{O}$ generated by a pair of octonions is associative. Algebras of this type are called alternative. Octonions can be obtained from quaternions via the Cayley–Dickson construction. It defines octonions as pairs of quaternions with multiplication

$$(a, b)(c, d) = (ac - d^* b, da + bc^*)$$

and conjugations

$$(a, b)^* = (a^*, -b).$$

Similarly to complex numbers and quaternions, octonions satisfy

$$(xy)^* = y^* x^*, \quad \forall_{z \neq 0} : |z| \equiv zz^* \in \mathbb{R}_+$$

These makes octonions a normed division algebra.

$$|xy| = |x|\,|y|\,, \quad \forall_{z \neq 0} : \ \exists z^{-1} = \frac{z^*}{|z|}.$$

This is the last available finite-dimensional normed division algebra.

Octonion Projective Plane

Similarly to the $2D$ and $4D$ models, the $8D$ conformal symmetries can be described as the symmetries of the octonion projective line $\mathbb{OP}^1$. Associate $\mathbb{T}$ with a space of 2×2 hermitian octonionic matrices

$$\mathbb{T} = \mathfrak{h}_2(\mathbb{O}) \ni h = \begin{pmatrix} h_{11} & h_{12} \\ h_{12}^* & h_{22} \end{pmatrix} \quad (h_{11} \in \mathbb{R},\ h_{22} \in \mathbb{R},\ h_{12} \in \mathbb{O})$$

with the hyperbolic quadratic form

$$(h, h) \equiv -\det(h) = |h_{12}|^2 - h_{11} h_{22}.$$

Light cone vectors are associated with singular matrices, which can be represented as

$$h = \begin{pmatrix} 1 \\ z \end{pmatrix} \otimes \begin{pmatrix} 1 & \bar{z} \end{pmatrix} (z \in \mathbb{O}).$$

The model can be extended by replacing $\mathbb{OP}^1$ with the octonionic (Cayley) projective plane $\mathbb{OP}^2$. However, due to the lack of associativity, higher dimensional extensions are not possible.

The $\mathbb{OP}^2$ Model: State Space And Wealth Bundle

Due to the lack of associativity, the standard definition of projective planes is not directly applicable. Instead, $\mathbb{OP}^2$ can be defined in terms of the space of octonionic hermitian matrices $\mathfrak{h}_3(\mathbb{O})$ ($\dim_{\mathbb{R}} \mathfrak{h}_3(\mathbb{O}) = 27$). Consider its subset consisting of rank 1 matrices and define $\mathbb{OP}^2$ as an associated projective space:

$$\mathfrak{h}_3^0(\mathbb{O}) = \left\{ h \in \mathfrak{h}_3^0(\mathbb{O}) \middle| \, \mathrm{rank}(h) = 1 \right\}, \quad \dim_{\mathbb{R}} \mathfrak{h}_3^0(\mathbb{O}) = 17, \quad \mathbb{OP}^2 = \mathfrak{h}_3^0(\mathbb{O})/(\mathbb{R}\cdot).$$

The wealth bundle can be defined via the projection $\mathfrak{h}_3^0(\mathbb{O}) \to \mathbb{OP}^2$. Namely, asset prices are linked to homogeneous function in $\mathfrak{h}_3^0(\mathbb{O})$ of the order $\gamma = 1 - \frac{8}{2} = -3$:

$$\mathfrak{h}_3^0(\mathbb{O}) \xrightarrow{\psi} \mathbb{R}, \quad \psi(\lambda h) = |\lambda|^{-3} \psi(\lambda h) \quad \left(\lambda \in \mathbb{R} \setminus \{0\}, \ h \in \mathfrak{h}_3^0(\mathbb{O})\right)$$

Associate single cashflows with the homogeneous functions of the form

$$\mathcal{B}_T(h) = \langle b_T, h \rangle^{-3} \quad \left(h \in \mathfrak{h}_3^0, b_T \in \bar{\mathfrak{h}}_3^0\right).$$

Here, $\bar{\mathfrak{h}}_3^0(\mathbb{O})$ denotes the duel counterpart of $\mathfrak{h}_3^0(\mathbb{O})$.

The $\mathbb{OP}^2$ Model: Symmetries, Discount Curve, and Dynamics

Symmetries of $\mathbb{OP}^2$ form the group $E_{6(-26)}$. This is a non-compact real form of the exception group E_6 ($\dim E_6 = 78$). To enable time homogeneity, b_T matrices should be related via a $E_{6(-26)}$ evolution:

$$b_T = e^{k(T-t)} U_{T-t} b_t \quad \left(U_\tau = e^{\epsilon \tau},\ \epsilon \in \mathfrak{e}_{6(-26)},\ k \in \mathbb{R} \right).$$

Here, $\mathfrak{e}_{6(-26)}$ denotes a Lie algebra of $E_6(-26)$. Select the instant cash associated matrix as

$$b_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{h}_3^0(\mathbb{O}).$$

The model state $h \in \mathbb{OP}^2$ ($\langle b_t, h \rangle \neq 0$) can be parameterized by a pair of octonions:

$$h = \left[\begin{pmatrix} 1 \\ z^1 \\ z^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & \bar{z}^1 & \bar{z}^2 \end{pmatrix} \right] \quad \left(z^1, z^2 \in \mathbb{O} \right).$$

In this parametrization, the unspanned diffusion generator is governed by a $2D$ octonionic hermitian covariation matrix $\sigma \in \mathfrak{h}(2, \mathbb{O})$, $\dim_{\mathbb{R}} \mathfrak{h}(2, \mathbb{O}) = 10$. The dimension of all the $16D$ symmetric matrices is 115.

Conclusions

- We have demonstrated that $2D$, $4D$, and $8D$ conformal unspanned volatility models can be extended. Under the extension, the model dimensions are multiplied by an integer factor. The extensions are related to the normed devision algebras $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, respectively.
- Extensions of the $2D$ and $4D$ models form infinite series'. However, the $8D$ model has a single 16-dimensional extension.
- In the extended models, besides the common volatility scale, some individual volatilities and correlations are unspanned separately.
- The extension eliminates certain shortcomings of the original models, particularly those manifested in the low rate regime. It also enables incremental model refinement.
- The extension can serve as a unified parsimonious multi-currency model, incorporating stochastic interest rates, foreign exchange rates, and volatilities.

Other Thoughts

We have demonstrated how to build partially solvable unspanned models with normed division algebras $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. The most fundamental normed devision algebra $\mathbb{R}$ has been, so far, left out. Is there a similar class of models associated with $\mathbb{R}$? Can it be the Black-Scholes model, or, more generally, a Gaussian class of models? Gaussian models are fully solvable in terms of Green functions.

In addition to obvious time and state translations, Gaussian models are symmetric under more exotic Appel transformations which combine changes in state variables with changes in the numeraire. For a $1D$ Gaussian model, the aggregated symmetry group is isomorphic to $\mathbb{R}^2 \rtimes SL(2,\mathbb{R})$, which is a subgroup of $SL(3,\mathbb{R})$ stabilizing a vector in $\mathbb{R}^3$. The latter is the symmetry group of the real projective plane $\mathbb{RP}^2$. In the context of the Gaussian model, only one of $\mathbb{RP}^2$ dimensions is used to describe the model state; the other is associated with time.

It might be possible to apply the similar interpretation to projective spaces over other norm devision algebras. This would generalize the Black-Scholes model. The generalized models would incorporate stochastic interest rates and volatilities while preserving tractability.

References

For more details on financial applications of conformal symmetries, please refer to:

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- Heymann, Matthias (2018). *The Adaptive Curve Evolution Model for Interest Rates.*

For further reading on abstract algebra and projective geometry, please refer to:

- Baez, John C. (2002). *The Octonions.* Bulletin of the American Mathematical Society. 39 (2)
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Acknowledgments

The author is grateful to Peter Carr and Santiago Garcia for continuous support and very valuable discussions.

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